Math 10A
Worksheet, Discussion #33; Wednesday, 8/1/2018
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1 Gaussian Elimination

1.1 Concepts

1. In order to solve a system of equations to find the solution or determine if there are zero or infinitely many solutions, use Gaussian elimination on the **augmented matrix**, a matrix formed by appending the answer vector to the original matrix. A system of equations is **consistent** if there is at least one solution and **inconsistent** if there are no solutions.

1.2 Problems

2. True **FALSE** As soon as we see a row like $(000\dots0|0)$ during Gaussian elimination, we know that the system will have infinitely many solutions.

Solution: See Problem 5.

3. **TRUE** False If we see a row like $(000\dots0|0)$ then we know the determinant of the matrix.

Solution: The determinant will have to be 0 because there will be 0 or ∞ solutions.

4. Use Gaussian elimination on the following augmented matrix. Write the equations these correspond to.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & 2 \\ -3 & 0 & -2 & -1 \end{array}\right)$$

Solution: Add 3 times the first row to the third row to get

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & 2 \\ -3 & 0 & -2 & -1 \end{array}\right) \xrightarrow{III+3I} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & 2 \\ 0 & 6 & 1 & 8 \end{array}\right) \xrightarrow{I+2II,III+6II} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -5 & 20 \end{array}\right)$$

$$\xrightarrow{III/-5} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{I+III, II+III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{-II} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

Therefore the solution is $(3, 2, -4)$.

5. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} 2x_1 + x_2 - x_3 = 4 \\ -4x_1 - 2x_2 + 2x_3 = -6 \\ 6x_1 + 3x_2 - 3x_3 = 12 \end{cases}$$

Solution: Writing this as an augmented matrix, we get

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ -4 & -2 & 2 & -6 \\ 6 & 3 & -3 & 12 \end{array}\right) \xrightarrow{II+2I, III-3I} = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So this system of equations has no solutions.

6. Find conditions on a, b such that the following system has no solutions, infinitely many, and a unique solution.

$$\begin{cases} x + ay = 2 \\ 4x + 8y = b \end{cases}$$

Solution: We want to solve the equation

$$\begin{pmatrix} 1 & a \\ 4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ b \end{pmatrix}.$$

We know that this has a unique solution if the determinant is nonzero so we need $8 - 4a \neq 0$ or $a \neq 2$. For all $a \neq 2$ and any b, this has a unique solution.

Now if $a = 2$, we know that this solution has zero or infinite solutions. To tell, we need to use Gaussian elimination. Putting it into an augmented matrix and solving gives us

$$\left(\begin{array}{cc|c} 1 & a & 2 \\ 4 & 8 & b \end{array}\right) \xrightarrow{II-4I} \left(\begin{array}{cc|c} 1 & a & 2 \\ 0 & 0 & b-8 \end{array}\right)$$

Thus if $b \neq 8$, then we have an inconsistent system and the system has no solutions. If $b = 8$, then there are infinitely many solutions.

7. Find $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 2 \end{pmatrix}^{-1}$.

Solution: We need to use Gaussian elimination to reduce

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{III-2I} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I-3II,III+II} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right) \xrightarrow{I+2III,II-III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

Thus the inverse is $\begin{pmatrix} -3 & -1 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}$.

8. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 5 \\ 2x_1 + 4x_2 + 12x_3 = -6 \\ x_1 - 4x_2 - 12x_3 = 9 \end{cases}$$

Solution: Using Gaussian elimination gives

$$\left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 2 & 4 & 12 & -6 \\ 1 & -4 & -12 & 9 \end{array}\right) \xrightarrow{II-2I,III-I} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 8 & 24 & -16 \\ 0 & -2 & -6 & 4 \end{array}\right)$$

$$\xrightarrow{II/8,III/-2} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{array}\right) \xrightarrow{I+2II,III-II} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solving, we get $x_1 = 1, x_2 = -2 - 3x_3$ and x_3 can be whatever it wants. This means there are an infinite number of solutions, that the determinant of the original matrix is 0.

9. Come up with an example of a consistent system of equations with 3 equations and 2 variables. Give an example of an inconsistent system of linear equations with 2 equations and 3 variables.

Solution: One example for a consistent system is $x+y=11, 2x+3y=23, -x+2y=-8$ with solution $x=10, y=1$.

One example for an inconsistent system is $x+2y+3z=2, x+2y+3z=1$, because it has no solutions.

10. Use Gaussian elimination to solve the following system of equations:

$$\begin{cases} z-3y=-6 \\ x-2y-2z=-14 \\ 4y-x-3z=5 \end{cases}$$

Solution: First write it in an augmented matrix.

$$\left(\begin{array}{ccc|c} 0 & -3 & 1 & -6 \\ 1 & -2 & -2 & -14 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{I\leftrightarrow II} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ -1 & 4 & -3 & 5 \end{array}\right) \xrightarrow{III+I} \left(\begin{array}{ccc|c} 1 & -2 & -2 & -14 \\ 0 & -3 & 1 & -6 \\ 0 & 2 & -5 & -9 \end{array}\right)$$

$$\xrightarrow{I-2/3II, III+2/3II} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & -13/3 & -13 \end{array}\right) \xrightarrow{III*3/-13} \left(\begin{array}{ccc|c} 1 & 0 & -8/3 & -10 \\ 0 & -3 & 1 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{I+8/3III, II-III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{II/-3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Thus the solution is $(-2, 3, 3)$.